Problem 1.32

If you have some experience in electromagnetism, you could do the following problem concerning the curious situation illustrated in Figure 1.8. The electric and magnetic fields at a point $\mathbf{r}_1$ due to a charge q_2 at $\mathbf{r}_2$ moving with constant velocity $\mathbf{v}_2$ (with $v_2 \ll c$) are[15]

$$\mathbf{E}(\mathbf{r}_1) = \frac{1}{4\pi\epsilon_o}\frac{q_2}{s^2}\hat{\mathbf{s}} \quad \text{and} \quad \mathbf{B}(\mathbf{r}_1) = \frac{\mu_o}{4\pi}\frac{q_2}{s^2}\mathbf{v}_2 \times \hat{\mathbf{s}}$$

where $\mathbf{s} = \mathbf{r}_1 - \mathbf{r}_2$ is the vector pointing from $\mathbf{r}_2$ to $\mathbf{r}_1$. (The first of these you should recognize as Coulomb's law.) If $\mathbf{F}_{12}^{\text{el}}$ and $\mathbf{F}_{12}^{\text{mag}}$ denote the electric and magnetic forces on a charge q_1 at $\mathbf{r}_1$ with velocity $\mathbf{v}_1$, show that $F_{12}^{\text{mag}} \le (v_1 v_2/c^2)F_{12}^{\text{el}}$. This shows that in the non-relativistic domain it is legitimate to ignore the magnetic force between two moving charges.

Solution

According to Coulomb's law, the electric force acting on q_1 is the charge times the electric field at its location.

$$\begin{aligned}\mathbf{F}_{12}^{\text{el}} &= q_1\mathbf{E}(\mathbf{r}_1) \\ &= q_1\left(\frac{1}{4\pi\epsilon_o}\frac{q_2}{s^2}\hat{\mathbf{s}}\right) \\ &= \frac{1}{4\pi\epsilon_o}\frac{q_1 q_2}{s^2}\hat{\mathbf{s}}\end{aligned}$$

The magnitude of this vector is

$$|\mathbf{F}_{12}^{\text{el}}| = F_{12}^{\text{el}} = \frac{1}{4\pi\epsilon_o}\frac{q_1 q_2}{s^2}.$$

Multiply both sides by ϵ_o.

$$\epsilon_o F_{12}^{\text{el}} = \frac{1}{4\pi}\frac{q_1 q_2}{s^2}$$

According to the Biot-Savart law, the magnetic force acting on q_1 is the charge times the cross product of its velocity and the magnetic field at its location.

$$\begin{aligned}\mathbf{F}_{12}^{\text{mag}} &= q_1\mathbf{v}_1 \times \mathbf{B}(\mathbf{r}_1) \\ &= q_1\mathbf{v}_1 \times \left(\frac{\mu_o}{4\pi}\frac{q_2}{s^2}\mathbf{v}_2 \times \hat{\mathbf{s}}\right) \\ &= \frac{\mu_o}{4\pi}\frac{q_1 q_2}{s^2}\mathbf{v}_1 \times (\mathbf{v}_2 \times \hat{\mathbf{s}}) \\ &= \mu_o(\epsilon_o F_{12}^{\text{el}})\mathbf{v}_1 \times (\mathbf{v}_2 \times \hat{\mathbf{s}}) \\ &= \frac{F_{12}^{\text{el}}}{c^2}\mathbf{v}_1 \times (\mathbf{v}_2 \times \hat{\mathbf{s}})\end{aligned}$$

[15] See, for example, David J. Griffiths, *Introduction to Electrodynamics*, 3rd ed., Prentice Hall, (1999), p. 440.

Recall that the magnitude of the cross product between two vectors, $\mathbf{A}$ and $\mathbf{B}$, is $|\mathbf{A}\times\mathbf{B}| = |\mathbf{A}||\mathbf{B}|\sin\beta$, where β is the angle between the vectors. Let θ be the angle between $\mathbf{v}_1$ and $\mathbf{v}_2\times\hat{\mathbf{s}}$, and let α be the angle between $\mathbf{v}_2$ and $\hat{\mathbf{s}}$. Then the magnitude of the magnetic force acting on q_1 is

$$\begin{aligned}
|\mathbf{F}_{12}^{\text{mag}}| = F_{12}^{\text{mag}} &= \frac{F_{12}^{\text{el}}}{c^2}|\mathbf{v}_1\times(\mathbf{v}_2\times\hat{\mathbf{s}})| \\
&= \frac{F_{12}^{\text{el}}}{c^2}v_1|\mathbf{v}_2\times\hat{\mathbf{s}}|\sin\theta \\
&= \frac{F_{12}^{\text{el}}}{c^2}v_1(v_2|\hat{\mathbf{s}}|\sin\alpha)\sin\theta \\
&= \left(\frac{v_1v_2}{c^2}\right)F_{12}^{\text{el}}\sin\alpha\sin\theta \\
&\le \left(\frac{v_1v_2}{c^2}\right)F_{12}^{\text{el}}.
\end{aligned}$$

Note that the magnitude of a unit vector is 1: $|\hat{\mathbf{s}}| = 1$. Also, $\sin\alpha\le 1$ and $\sin\theta\le 1$.